On the Admissibility of a Polish Group Topology and Other Things

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The Beginning of the Story

Question (Evans)

Can an uncountable free group be the automorphism group of a countable structure?

Answer (Shelah [Sh:744])

[1] *No uncountable free group can be the group of automorphisms of a countable structure.*

[1] S. Shelah. *A Countable Structure Does Not Have a Free Uncountable Automorphism Group*. Bull. London Math. Soc. **35** (2003), 1-7.

Polish Groups

Question (Becker and Kechris)

Can an uncountable free group admit a Polish group topology?

Answer (Shelah [Sh:771])

[2] *No uncountable free group can admit a Polish group topology.*

[2] Saharon Shelah. *Polish Algebras, Shy From Freedom*. Israel J. Math. **181** (2011), 477-507.

Some History/Literature

The question above was answered by Dudley[3] "before the it was asked". In fact Dudley proved a more general result, but with techniques very different from Shelah's.

Inspired by the above question of Becker and Kechris, Solecki[4] proved that no uncountable Polish group can be free abelian. Also Solecki's proof used methods very different from Shelah's.

[3] Richard M. Dudley. *Continuity of Homomorphisms*. Duke Math. J. **28** (1961), 587-594.

[4] Sławomir Solecki. *Polish Group Topologies*. In: Sets and Proofs, London Math. Soc. Lecture Note Ser. 258. Cambridge University Press, 1999.

The Completeness Lemma for Polish Groups

The crucial technical tool used by Shelah in his proof is what he calls a Completeness (or Compactness) Lemma for Polish Groups.

This is a technical result stating that if G is a Polish group, then for every sequence $\bar{d} = (d_n : n < \omega) \in G^\omega$ converging to the identity element $e_G = e$, many countable sets of equations with parameters from $\bar{d}$ are solvable in G.

Using The Completeness Lemma

The aim of our work was to extend the scope of applications of the techniques from [Sh:771] to other classes of groups from combinatorial and geometric group theory, most notably:

- right-angled Artin and Coxeter groups;
- graph products of cyclic groups;
- graph products of groups.

Our Papers (G.P. and Saharon Shelah)

- *No Uncountable Polish Group Can be a Right-Angled Artin Group*. Axioms **6** (2017), no. 2: 13.
- *Polish Topologies for Graph Products of Cyclic Groups*. Israel J. Math., to appear.
- *Group Metrics for Graph Products of Cyclic Groups*. Topology Appl. **232** (2017), 281-287.
- *Polish Topologies for Graph Products of Groups*. Submitted.

Right-Angled Artin Groups

Definition

Given a graph $\Gamma = (E, V)$, the associated right-angled Artin group (a.k.a RAAG) $A(\Gamma)$ is the group with presentation:

$$\Omega(\Gamma) = \langle V \mid ab = ba : aEb \rangle.$$

If in the presentation $\Omega(\Gamma)$ we ask in addition that all the generators have order 2, then we speak of the right-angled Coxeter group (a.k.a RACG) $C(\Gamma)$.

Examples

Let Γ_1 be a discrete graph (no edges), then $A(\Gamma_1)$ is a free group.

Let Γ_2 be a complete graph (a.k.a. clique), then $A(\Gamma_2)$ is a free abelian group, and $C(\Gamma_2)$ is the abelian group $\bigoplus_{\alpha<|\Gamma|} \mathbb{Z}_2$.

No Uncountable Polish group can be a RAAG

Theorem (P. & Shelah)

Let $G = (G, \mathfrak{d})$ be an uncountable Polish group and A a group admitting a system of generators whose associated length function satisfies the following conditions:

(i) *if $0 < k < \omega$, then $lg(x) \leqslant lg(x^k)$;*

(ii) *if $lg(y) < k < \omega$ and $x^k = y$, then $x = e$.*

Then G is not isomorphic to A, in fact there exists a subgroup G^* of G of size $\mathfrak{b}$ (the bounding number) such that G^* is not embeddable in A.

Corollary (P. & Shelah)

No uncountable Polish group can be a right-angled Artin group.

What about right-angled Coxeter groups?

The structure M with ω many disjoint unary predicates of size 2 is such that $Aut(M) = (\mathbb{Z}_2)^\omega = \bigoplus_{\alpha < 2^\omega} \mathbb{Z}_2$, i.e. $Aut(M)$ is the right-angled Coxeter group on the complete graph $K_{2^{\aleph_0}}$.

Question

Which right-angled Coxeter groups admit a Polish group topology (resp. a non-Archimedean Polish group topology)?

Graph Products of Cyclic Groups

Definition

Let $\Gamma = (V, E)$ *be a graph and let:*

$$\mathfrak{p} : V \to \{p^n : p \text{ prime and } 1 \leqslant n\} \cup \{\infty\}$$

a vertex graph coloring (i.e. $\mathfrak{p}$ *is a function). We define a group* $G(\Gamma, \mathfrak{p})$ *with the following presentation:*

$$\langle V \mid a^{\mathfrak{p}(a)} = 1,\ bc = cb : \mathfrak{p}(a) \neq \infty \text{ and } bEc\rangle.$$

Examples

Let $(\Gamma, \mathfrak{p})$ be as above and suppose that $ran(\mathfrak{p}) = \{\infty\}$, then $G(\Gamma, \mathfrak{p})$ is a right-angled Artin group.

Let $(\Gamma, \mathfrak{p})$ be as above and suppose that $ran(\mathfrak{p}) = \{2\}$, then $G(\Gamma, \mathfrak{p})$ is a right-angled Coxeter group.

A Characterization

Theorem (P. & Shelah)

Let $G = G(\Gamma, \mathfrak{p})$. Then G admits a Polish group topology if only if $(\Gamma, \mathfrak{p})$ satisfies the following four conditions:

(a) *there exists a countable $A \subseteq \Gamma$ such that for every $a \in \Gamma$ and $a \neq b \in \Gamma - A$, a is adjacent to b;*

(b) *there are only finitely many colors c such that the set of vertices of color c is uncountable;*

(c) *there are only countably many vertices of color ∞;*

(d) *if there are uncountably many vertices of color c, then the set of vertices of color c has the size of the continuum.*

Furthermore, if $(\Gamma, \mathfrak{p})$ satisfies conditions (a)-(d) above, then G can be realized as the group of automorphisms of a countable structure.

In Plain Words

Theorem (P. & Shelah)

The only graph products of cyclic groups $G(\Gamma, \mathfrak{p})$ *admitting a Polish group topology are the direct sums* $G_1 \oplus G_2$ *with* G_1 *a countable graph product of cyclic groups and* G_2 *a direct sum of finitely many continuum sized vector spaces over a finite field.*

Embeddability of Graph Products into Polish groups

Fact

The free group on continuum many generators is embeddable into $Sym(\omega)$ (the symmetric group on a countably infinite set).

Question

Which graph products of cyclic groups $G(\Gamma, \mathfrak{p})$ are embeddable into a Polish group?

Another Characterization

Theorem (P. & Shelah)

Let $G = G(\Gamma, \mathfrak{p})$, then the following are equivalent:

(a) *there is a metric on Γ which induces a separable topology in which E_Γ is closed;*

(b) *G is embeddable into a Polish group;*

(c) *G is embeddable into a non-Archimedean Polish group.*

The condition(s) above fail e.g. for the $\aleph_1$-half graph $\Gamma = \Gamma(\aleph_1)$, i.e. the graph on vertex set $\{a_\alpha : \alpha < \aleph_1\} \cup \{b_\beta : \beta < \aleph_1\}$ with edge relation defined as $a_\alpha E_\Gamma b_\beta$ if and only if $\alpha < \beta$.

Even More...

Theorem (P. & Shelah)

Let $\Gamma = (\omega^\omega, E)$ *be a graph and*

$$\mathfrak{p} : V \to \{p^n : p \text{ prime}, n \geqslant 1\} \cup \{\infty\}$$

a vertex graph coloring. Suppose further that E *is closed in the Baire space* ω^ω*, and that* $\mathfrak{p}(\eta)$ *depends*[5] *only on* $\eta(0)$*. Then* $G = G(\Gamma, \mathfrak{p})$ *admits a left-invariant separable group ultrametric extending the standard metric on the Baire space.*

This generalizes results of Gao et al. on left-invariant group metrics on free groups on continuum many generators.

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[5] I.e., for $\eta, \eta' \in 2^\omega$, we have: $\eta(0) = \eta'(0)$ implies $\mathfrak{p}(\eta) = \mathfrak{p}(\eta')$. This is essentially a technical convenience.

The Last Level of Generality

Definition

Let $\Gamma = (V, E)$ be a graph and $\{G_a : a \in \Gamma\}$ a set of non-trivial groups each presented with its multiplication table presentation and such that for $a \neq b \in \Gamma$ we have $e_{G_a} = e = e_{G_b}$ and $G_a \cap G_b = \{e\}$. We define the graph product of the groups $\{G_a : a \in \Gamma\}$ over Γ, denoted $G(\Gamma, G_a)$, via the following presentation:

$$\text{generators: } \bigcup_{a \in V} \{g : g \in G_a\},$$

$$\text{relations: } \bigcup_{a \in V} \{\text{the relations for } G_a\}$$

$$\cup \bigcup_{\{a,b\} \in E} \{gg' = g'g : g \in G_a \text{ and } g' \in G_b\}.$$

Examples

Let Γ be a graph and let, for $a \in \Gamma$, G_a be a primitive[6] cyclic group. Then $G(\Gamma, G_a)$ is a graph product of cyclic groups $G(\Gamma, \mathfrak{p})$.

[6] I.e. a cyclic group of order of the form p^n or infinity.

Some Notation

Notation

(1) *We denote by* $G^*_\infty = \mathbb{Q}$ *the rational numbers, by* $G^*_p = \mathbb{Z}^\infty_p$ *the divisible abelian p-group of rank* 1 *(Prüfer p-group), and by* $G^*_{(p,k)} = \mathbb{Z}_{p^k}$ *the finite cyclic group of order* p^k*.*

(2) *We let* $S_* = \{(p,k) : p \text{ prime and } k \geqslant 1\} \cup \{\infty\}$ *and* $S_{**} = S_* \cup \{p : p \text{ prime}\}$*;*

(3) *For* $s \in S_{**}$ *and* λ *a cardinal, we let* $G^*_{s,\lambda}$ *be the direct sum of* λ *copies of* G^*_s*.*

The First Venue

Theorem (P. & Shelah)

Let $G = G(\Gamma, G_a)$ and suppose that G admits a Polish group topology. Then for some countable $A \subseteq \Gamma$ and $1 \leqslant n < \omega$ we have:

(a) *for every $a \in \Gamma$ and $a \neq b \in \Gamma - A$, a is adjacent to b;*

(b) *if $a \in \Gamma - A$, then $G_a = \bigoplus \{ G^*_{s, \lambda_{a,s}} : s \in S_* \}$;*

(c) *if $\lambda_{a,(p,k)} > 0$, then $p^k \mid n$;*

(d) *if in addition $A = \emptyset$, then for every $s \in S_*$ we have that $\sum \{ \lambda_{a,s} : a \in \Gamma \}$ is either $\leqslant \aleph_0$ or $2^{\aleph_0}$.*

The Second Venue

Theorem (P. & Shelah)

Let $G = G(\Gamma, G_a)$*. Then there is a finite subset* B_1 *of* Γ *such that if we let* $B = \Gamma - B_1$ *then the following conditions are equivalent:*

(a) $G(\Gamma \restriction B)$ *admits a Polish group topology;*

(b) *for every* $s \in S_*$ *the cardinal:*

$$\lambda_s^B = \sum \{\lambda_{a,s} : a \in B\} \in \{\aleph_0, 2^{\aleph_0}\}.$$

The Third Venue

Corollary (P. & Shelah)

Let $G = G(\Gamma, G_a)$ with all the G_a countable. Then G admits a Polish group topology if only if G admits a non-Archimedean Polish group topology if and only if there exist a countable $A \subseteq \Gamma$ and $1 \leqslant n < \omega$ such that:

(a) *for every $a \in \Gamma$ and $a \neq b \in \Gamma - A$, a is adjacent to b;*

(b) *if $a \in \Gamma - A$, then $G_a = \bigoplus\{G^*_{s,\lambda_{a,s}} : s \in S_*\}$;*

(c) *if $\lambda_{a,(p,k)} > 0$, then $p^k \mid n$;*

(d) *for every $s \in S_*$, $\sum\{\lambda_{a,s} : a \in \Gamma - A\}$ is either $\leqslant \aleph_0$ or $2^{\aleph_0}$.*

The Third Venue (Cont.)

Corollary (P. & Shelah)

Let G be an abelian group which is a direct sum of countable groups, then G admits a Polish group topology if only if G admits a non-Archimedean Polish group topology if and only if there exists a countable $H \leqslant G$ and $1 \leqslant n < \omega$ such that:

$$G = H \oplus \bigoplus_{\alpha < \lambda_\infty} \mathbb{Q} \oplus \bigoplus_{p^k | n} \bigoplus_{\alpha < \lambda_{(p,k)}} \mathbb{Z}_{p^k},$$

with λ_∞ and $\lambda_{(p,k)} \leqslant \aleph_0$ or $2^{\aleph_0}$.

Corollary (P. & Shelah, and independently Slutsky)

If G is an uncountable group admitting a Polish group topology, then G can not be expressed as a non-trivial free product.

A Conjecture

Conjecture (Polish Direct Summand Conjecture)

Let G be a group admitting a Polish group topology.

(1) *If G has a direct summand isomorphic to $G^*_{s,\lambda}$, for some $\aleph_0 < \lambda \leqslant 2^{\aleph_0}$ and $s \in S_*$, then it has one of cardinality $2^{\aleph_0}$.*

(2) *If $G = G_1 \oplus G_2$ and $G_2 = \bigoplus\{G^*_{s,\lambda_s} : s \in S_*\}$, then for some G'_1, G'_2 we have:*
   - (i) $G_1 = G'_1 \oplus G'_2$;
   - (ii) G'_1 *admits a Polish group topology;*
   - (iii) $G'_2 = \bigoplus\{G^*_{s,\lambda'_s} : s \in S_*\}$.

(3) *If $G = G_1 \oplus G_2$, then for some G'_1, G'_2 we have:*
   - (i) $G_1 = G'_1 \oplus G'_2$;
   - (ii) G'_1 *admits a Polish group topology;*
   - (iii) $G'_2 = \bigoplus\{G^*_{s,\lambda_s} : s \in S_*\}$.

Reconstruction Theory

Reconstruction theory deals with the problem of reconstruction of model-theoretic properties of a (countable) structure M from a naturally associated algebraic or topological object $X(M)$, e.g. its automorphism group or its endomorphism monoid.

Prototipical results are:

$$X(M) \sim_1 X(M') \text{ if and only if } M \sim_2 M',$$

for given equivalence relations $\sim_1$ and $\sim_2$ of interest.

Techniques and Degrees of Reconstruction

Two independent techniques lead the scene in this field:

(A) the (strong) small index property;

(B) Rubin's $\forall\exists$-interpretation.

Furthermore, there are two classical degrees of reconstruction:

1. reconstruction up to bi-interpretability;
2. reconstruction up to bi-definability.

These correspond to:

1. abstract group $\cong \implies$ topological group $\cong$;
2. abstract group $\cong \implies$ permutation group $\cong$.

The Strong Small Index Property

Let M be a countable structure.

- Small index property (SIP): every subgroup of $Aut(M)$ of index less than the continuum contains the pointwise stabilizer of a finite set $A \subseteq M$.
- Strong small index property (SSIP): every subgroup of $Aut(M)$ of index less than the continuum lies between the pointwise and the setwise stabilizer of a finite set $A \subseteq M$.

State of the Art on First Degree of Reconstruction

Theorem (Rubin)

Let M and N be countable $\aleph_0$-categorical structures and suppose that M has a $\forall\exists$-interpretation. Then $Aut(M) \cong Aut(N)$ if and only if M and N are bi-interpretable.

Theorem (Lascar (based on works of Ahlbrandt and Ziegler))

Let M and N be countable $\aleph_0$-categorical structures and suppose that M has the small index property. Then $Aut(M) \cong Aut(N)$ if and only if M and N are bi-interpretable.

State of the Art on Second Degree of Reconstruction

Theorem (Rubin)

Let M and N be countable $\aleph_0$-categorical structures with no algebraicity and suppose that M has a $\forall\exists$-interpretation. Then $Aut(M) \cong Aut(N)$ if and only if M and N are bi-definable.

Theorem

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Algebraicity

Definition

Let M be a structure and $G = Aut(M)$.

(1) *We say that a is algebraic over $A \subseteq M$ in M if the orbit of a under $G_{(A)}$ is finite.*

(2) *The* algebraic closure *of $A \subseteq M$ in M, denoted as $acl_M(A)$, is the set of elements of M which are algebraic over A.*

The Missing Piece

Theorem (P. & Shelah)

Let $\mathbf{K}_*$ *be the class of countable structures* M *satisfying:*

(1) M *has the strong small index property;*

(2) *for every finite* $A \subseteq M$, $acl_M(A)$ *is finite;*

(3) *for every* $a \in M$, $acl_M(\{a\}) = \{a\}$.

Then for $M, N \in \mathbf{K}_*$, $Aut(M)$ *and* $Aut(N)$ *are isomorphic as abstract groups if and only if* $(Aut(M), M)$ *and* $(Aut(N), N)$ *are isomorphic as permutation groups.*

The Missing Piece (Cont.)

Corollary (P. & Shelah)

Let M and N be countable $\aleph_0$-categorical structures with the strong small index property and no algebraicity. Then $Aut(M)$ and $Aut(N)$ are isomorphic as abstract groups if and only if M and N are bi-definable. Furthermore, if $f : M \to N$ witnesses the bi-definability of M and N, then f induces the isomorphism of abstract groups $\pi_f : Aut(M) \cong Aut(N)$ given by $\alpha \mapsto f\alpha f^{-1}$.

Outer Automorphisms Groups

Given a group G we let:

$$Out(G) = Aut(G)/Inn(G).$$

We say that G is complete if it has trivial center and $Aut(G) = Inn(G)$ (which implies that $G \cong Aut(G)$).

If M satisfies the conclusion of the theorem above, then any $f \in Aut(Aut(M))$ is induced by a permutation of M.

With this it is easy to see:

(1) Letting R_n be the n-coloured random graph ($n \geqslant 2$) we have that $Out(Aut(R_n)) \cong Sym(n)$.

(2) Letting M_n be the K_n-free random graph ($n \geqslant 3$) we have that $Aut(M_n)$ is complete.

Outer Automorphisms Groups (Cont.)

Theorem (P. & Shelah)

Let K be a finite group. Then there exists a countable $\aleph_0$-categorical homogeneous structure M with the strong small index property and no algebraicity such that $K \cong Out(Aut(M))$.

Automorphism Groups and Model-Theoretic Stability

One of the main tools in modern model theory are certain dividing lines which classify theories in function of how much combinatorial information can be coded in the class of its models.

This area of research was invented by Shelah and has had huge number of applications in the most disparate fields of mathematics.

The most well-known dividing lines are: $\aleph_0$-stability, superstability, and stability. Each of these notions can be defined in terms of bounds on the number of types over sets of parameters for models of a given theory, but there are many equivalent definitions.

On a Problem of Rosendal

In a recent work Rosendal isolates a property of topological groups which he calls locally boundedness and proves that if M is the countable, saturated model of an $\aleph_0$-stable theory then $Aut(M)$ is locally bounded.

Rosendal asks if the property of locally boundedness is satisfied by the group of automorphisms of any countable model of an $\aleph_0$-stable theory. This was settled in the negative by Zielinski[7]

[7] Zielinski proved more, i.e. that the model can be taken to be atomic.

An Impossibility Result

Theorem (P. & Shelah)

For every countable structure M there exists an $\aleph_0$-stable countable structure N such that $Aut(M)$ and $Aut(N)$ are topologically isomorphic with respect to the naturally associated Polish group topologies.

This shows that it is impossible to detect any form of stability of a countable structure M from the topological properties of the Polish group $Aut(M)$.

Our Other Papers (G.P. and Saharon Shelah)

- *Reconstructing Structures with the Strong Small Index Property up to Bi-Definability*. Fund. Math., to appear.
- *The Automorphism Group of Hall's Universal Group*. Proc. Amer. Math. Soc. **146** (2018), 1439-1445.
- *The Strong Small Index Property for Free Homogeneous Structures*. Submitted.
- *Automorphism Groups of Countable Stable Structures*. Submitted.