Introduction to Permutation Models

with an Application to Ring Theory

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The Language of Set Theory with Atoms

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- The collection of atoms is denoted by A, and we add the constant symbol $\mathtt{A}$ to the language of Set Theory.
- The language of Set Theory with atoms, denoted ZFA, consists of the relation symbol "$\in$" and the constant symbol "$\mathtt{A}$".

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- **Axiom of Atoms**

$$\forall x\big(x \in \mathtt{A} \leftrightarrow \big(x \neq \emptyset \land \neg \exists z(z \in x)\big)\big)$$

A Model of ZFA + AC

$$\begin{aligned}
\mathrm{M}_0 &:= A\,, \\
\mathrm{M}_\alpha &:= \bigcup_{\beta\in\alpha} \mathrm{M}_\beta \quad \text{if } \alpha \text{ is a limit ordinal}\,, \\
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- $\hat{\mathbf{V}} := \bigcup_{\alpha\in\Omega} \mathscr{P}^\alpha(\emptyset)$ is a model of ZF and is called the **kernel** of $\boldsymbol{\mathcal{M}}$.
- If the construction of $\boldsymbol{\mathcal{M}}$ was carried out in a model of ZFC, then $\hat{\mathbf{V}} \models$ ZFC and $\boldsymbol{\mathcal{M}} \models$ ZFA + AC.

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  (C) if $\pi \in \mathcal{G}$ and $H \in \mathscr{F}$, then $\pi H \pi^{-1} \in \mathscr{F}$

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  (C) if $\pi \in \mathcal{G}$ and $H \in \mathscr{F}$, then $\pi H \pi^{-1} \in \mathscr{F}$
  (D) for each $a \in A$, $\{\pi \in \mathcal{G} : \pi a = a\} \in \mathscr{F}$

Permutation Models: a simple normal filter

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Then the filter $\mathscr{F}$ on $\mathcal{G}$ generated by the subgroups $\text{fix}_{\mathcal{G}}(E)$, where E is a finite subset of A, is a normal filter.

Permutation Models: symmetric sets

For every $\pi \in \mathcal{G}$ and for every set $x \in \boldsymbol{\mathcal{M}}$ we can define πx by stipulating

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For $x \in \mathcal{M}$, the **symmetry group** of x, denoted $\mathrm{sym}_{\mathcal{G}}(x)$, is defined by

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A set x is **symmetric (with respect to $\mathscr{F}$)** if

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- If $\mathscr{F}$ is the normal filter of finite sets, then a set x belongs to $\mathcal{V}$ if and only if there exists a *finite* set of atoms $E_x \subseteq A$, called a **support** of x, such that

$$\mathrm{fix}_{\mathcal{G}}(E_x) \subseteq \mathrm{sym}_{\mathcal{G}}(x).$$

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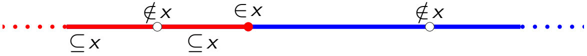

- For a given finite set $E \subset A$ of size m, there are 2^{2m+1} sets $x \subseteq A$ with support E.

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This gives us just consistency with ZFA, but fortunately, the result can be transferred to models of ZF!

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$\Rightarrow$ The existence of such a set X is independent of ZF.

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Question

For every positive integer n:

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Integral Domains

Pair Choice for Families of n-element Sets $2C_n$:
If $\mathscr{F} = \{Y_\lambda : \lambda \in \Lambda\}$ is a family of n-element, then from each $Y_\lambda \in \mathscr{F}$ we can choose a non-empty set with at most 2 elements.

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Proposition (H., H., L., L., L., S.)

For odd integers $n \geq 3$,

$$2C_n \Rightarrow \text{every integral domain has an } n\text{th-root function.}$$

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In order to show that the answer to the question is **no**, it is enough to prove $2C_n \not\Rightarrow C_n$ (for some odd integer n).

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For this, we modify Mostowski's model:

- Each atom a_i ($i \in \mathbb{Q}$) of Mostowski's model is replaced with two disjoint sets

$$Q_i = \{b_{i,0}, b_{i,1}\}$$

and

$$P_i = \{a_{i,0}, \ldots, a_{i,p-1}\},$$

where $p > 2$ is an arbitrary but fixed prime.

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- The set of atoms A:

$$A = \bigcup_{i \in \mathbb{Q}} Q_i \cup \bigcup_{j \in \mathbb{Q}} P_j.$$

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Let $\mathcal{G}$ be the group of permutations of A consisting of permutations π which satisfy

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Proposition (H., H., L., L., L., S.)

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Proof. In the modified Mostowski model we have

$$\neg \mathbf{C_{p+2}}$$

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$2\mathrm{C}_{p+2} \nRightarrow \mathrm{C}_{p+2}$ is consistent with ZF.

Proof. In the modified Mostowski model we have

$$\neg\mathbf{C}_{\mathbf{p+2}} \wedge \mathbf{2C}_{\mathbf{p+2}}.$$

⊣

$2C_n \not\Rightarrow C_n$

Corollary (H., H., L., L., L., S.)

every integral domain has an 5th-root function $\not\Rightarrow C_5$

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⊣