Generalised Closed Unbounded and Stationary Sets

Hazel Brickhill

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A sketch

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C is *1-club* in κ iff C is stationary in κ and stationary-closed.

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Notation

$d_\gamma(A) := \{\alpha : A \text{ is } \gamma\text{-stationary below } \alpha\}$

Restating the Definitions in Terms of d_γ

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Definition (restated)

1. $S \subseteq On$ is 0-stationary in κ if it is unbounded in κ.
2. $C \subseteq On$ is *γ-stationary closed* if $d_\gamma(C) \subseteq C$.
3. C is *γ-club* in κ if C is γ-stationary closed and γ-stationary below κ.
4. κ is *γ-s-reflecting* if for any γ-stationary S, $T \subseteq \kappa$, $d_\gamma(S) \cap d_\gamma(T) \cap \kappa \neq \emptyset$.
5. $S \subseteq \kappa$ is $n+1$-stationary if κ is n-reflecting and $S \cap C \neq \emptyset$ for every C n-club in κ

how large is a subset of κ?

If κ is *n*-reflecting, then for a subset of κ we have these implications:

$$\begin{array}{ccc}
n\text{-club} & \Longrightarrow & n+1\text{-stationary} \\
\Uparrow & & \Downarrow \\
n-1\text{-club} & & n\text{-stationary} \\
\Uparrow & & \Downarrow \\
\dots & & \dots \\
\Uparrow & & \Downarrow \\
0\text{-club } (= \text{club}) & & \text{stationary} \\
 & & \Downarrow \\
 & & \text{unbounded}
\end{array}$$

Origins

- Sun, W. (1993). *Stationary cardinals*. Archive for Mathematical Logic, 32(6), 429-442.
- Hellsten, A. (2003). *Diamonds on large cardinals* (Vol. 134). Suomalainen Tiedeakatemia.
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- Bagaria, J., Magidor, M., and Sakai, H. (2015) *Reflection and indescribability in the constructible universe.* Israel Journal of Mathematics 208.1: 1-11.
- Bagaria, J. (2016). *Derived topologies on ordinals and stationary reflection.* https://www.newton.ac.uk/files/preprints/ni16031.pdf

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Theorem (Magidor)

A regular cardinal that is 1-s-reflecting is Π^1_1-indescribable in L. Thus the existence of a 1-s-reflecting cardinal is equiconsistent with the existence of a Π^1_1-indescribable.

Results in L

Magidor's equiconsistency proof uses the following:

Theorem (Jensen) ($V = L$)

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Definition

A $\Box^\gamma$ *sequence* below κ is a sequence $\langle C_\alpha : \alpha \in d_\gamma(\kappa)\rangle$ so that for all α:

1. C_α is an γ-club subset of α
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Theorem (B.) ($V = L$)

If κ is Π^1_n- but not Π^1_{n+1}-indescribable then for any $n+1$-stationary $A \subseteq \kappa$ there is an $n+1$-stationary set $A' \subseteq A$ and $\square^n$ sequence avoiding A'. Thus κ is not $n+1$-reflecting. ($0 < n < \omega$)

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Definition

A $\square^{<\lambda}$ *sequence* below κ is a sequence $\langle (\eta_\alpha, C_\alpha) : \alpha \in \kappa \rangle$ such that for each α:

1. $\eta \in \lambda$ and C_α is an η_α-club subset of α
2. for every $\beta \in d_{\eta_\alpha}(C_\alpha)$ we have $\eta_\beta = \eta_\alpha$ and $C_\beta = C_\alpha \cap \beta$

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This proof extends to replacing n with $\gamma < \kappa$ but only for successor stages: we need a slightly different $\square$ sequence for the limit stages.

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Theorem (B.) ($V = L$)

For $\gamma < \kappa$, if κ is not Π^1_γ-indescribable but for all $\gamma' < \gamma$ we have $\Pi^1_{\gamma'}$-indescribable then for any γ-stationary $A \subseteq \kappa$ there is a γ-stationary set $A' \subseteq A$ and a $\square^{<\gamma}$ sequence avoiding A'.
Thus κ is not γ-reflecting.

Consistency Strength?

Theorem (Magidor)

A regular cardinal that is 1-s-reflecting is Π^1_1-indescribable in L. Thus the existence of a 1-s-reflecting cardinal is equiconsistent with the existence of a Π^1_1-indescribable.

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Theorem (B.)

Let κ *be a regular cardinal that is* γ*-s-reflecting such that the* γ*-club filter on* κ *is normal, and for* γ*-stationary many cardinals* λ *below* κ *we have* λ *is* η*-s-reflecting implies the* η*-club filter on* λ *is normal. Then* κ *is* Π^1_γ*-indescribable in L.*

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Theorem (B.)

Let κ be a regular cardinal that is γ-s-reflecting such that the γ-club filter on κ is normal, and for γ-stationary many cardinals λ below κ we have λ is η-s-reflecting implies the η-club filter on λ is normal. Then κ is Π^1_γ-indescribable in L.

Conjecture:
For $\gamma > 1$ the consistency strength of a γ-s-reflecting cardinal is below that of a Π^1_γ-indescribable.

Generalising $\square(\kappa)$

Definition

For $\gamma < \kappa$, $\square^\gamma(\kappa)$ holds if there is a $\square^\gamma$ sequence $\langle C_\alpha : \alpha \in d_\gamma(\kappa)\rangle$ that has no thread, i.e. there is no γ club $C \subseteq \kappa$ such that for every $\beta \in d_\gamma(C)$ we have $C_\beta = C \cap \beta$

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Theorem (B.)

Suppose κ is a regular γ-reflecting cardinal and the γ-club filter on κ is normal. Let $S = \langle C_\alpha : \alpha \in d_\gamma(\kappa)\rangle$ be a $\square^\gamma$ sequence. Then the following are equivalent:

1. *S is a $\square^\gamma(\kappa)$ sequence, i.e. S has no thread.*
2. *For any $\gamma+1$-stationary set T there are $\gamma+1$-stationary $S_0, S_1 \subseteq T$ such that for any $\alpha \in d_\gamma(\kappa)$ we have $d_\gamma(C_\alpha) \cap S_0 = \emptyset$ or $d_\gamma(C_\alpha) \cap S_1 = \emptyset$*